

# BETA-EXPANSION AND CONTINUED FRACTION EXPANSION OF REAL NUMBERS

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**ABSTRACT.** Let  $\beta > 1$  be a real number and  $x \in [0, 1)$  be an irrational number. We denote by  $k_n(x)$  the exact number of partial quotients in the continued fraction expansion of  $x$  given by the first  $n$  digits in the  $\beta$ -expansion of  $x$  ( $n \in \mathbb{N}$ ). It is known that  $k_n(x)/n$  converges to  $(6 \log 2 \log \beta)/\pi^2$  almost everywhere in the sense of Lebesgue measure. In this paper, we improve this result by proving that the Lebesgue measure of the set of  $x \in [0, 1)$  for which  $k_n(x)/n$  deviates away from  $(6 \log 2 \log \beta)/\pi^2$  decays to 0 exponentially as  $n$  tends to  $\infty$ , which generalizes the result of Faivre [8] from  $\beta = 10$  to any  $\beta > 1$ . Moreover, we also discuss which of the  $\beta$ -expansion and continued fraction expansion yields the better approximations of real numbers.

## 1. INTRODUCTION

Let  $\beta > 1$  be a real number and  $T_\beta : [0, 1) \rightarrow [0, 1)$  be the  $\beta$ -transformation defined as

$$T_\beta(x) = \beta x - [\beta x],$$

where  $[x]$  denotes the greatest integer not exceeding  $x$ . Then every  $x \in [0, 1)$  can be uniquely expanded into a finite or infinite series, i.e.,

$$x = \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \cdots + \frac{\varepsilon_n(x)}{\beta^n} + \cdots, \quad (1.1)$$

where  $\varepsilon_1(x) = [\beta x]$  and  $\varepsilon_{n+1}(x) = \varepsilon_1(T_\beta^n x)$  for all  $n \geq 1$ . We call the representation (1.1) the  $\beta$ -expansion of  $x$  denoted by  $(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x), \dots)$  and  $\varepsilon_n(x), n \geq 1$  the *digits* of the  $\beta$ -expansion of  $x$ . Such an expansion was first introduced by Rényi [26], who proved that there exists a unique  $T_\beta$ -invariant measure  $\mu$  equivalent to the Lebesgue measure  $\lambda$ . More precisely,

$$C^{-1}\lambda(A) \leq \mu(A) \leq C\lambda(A) \quad (1.2)$$

for any Borel set  $A \subseteq [0, 1)$ , where  $C > 1$  is a constant only depending on  $\beta$ . Furthermore, Gel'fond [14] and Parry [23] independently found the density formula for this invariant measure with respect to (w.r.t.) the Lebesgue measure. Philipp [24] showed that the dynamical system  $([0, 1), \mathcal{B}, T_\beta, \mu)$  is an exponentially strong mixing measure-preserving system, where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $[0, 1)$ . Later, Hofbauer [15] proved that  $\mu$  is the unique measure of maximal entropy for  $T_\beta$ . Some arithmetic and metric properties of  $\beta$ -expansion were studied in the literature, such as [1, 3, 4, 7, 12, 19, 27, 28] and the references therein.

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Now we turn to introducing continued fractions. Let  $T : [0, 1) \rightarrow [0, 1)$  be the *Gauss transformation* given by

$$T(0) := 0 \quad \text{and} \quad T(x) := 1/x - [1/x] \quad \text{if } x \in (0, 1).$$

Then any irrational number  $x \in [0, 1)$  can be written as

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{\ddots + \frac{1}{a_n(x) + \ddots}}}}, \quad (1.3)$$

where  $a_1(x) = [1/x]$  and  $a_{n+1}(x) = a_1(T^n x)$  for all  $n \geq 1$ . The form (1.3) is said to be the *continued fraction expansion* of  $x$  and  $a_n(x), n \geq 1$  are called the *partial quotients* of the continued fraction expansion of  $x$ . Sometimes we write the form (1.3) as  $[a_1(x), a_2(x), \dots, a_n(x), \dots]$ . For any  $n \geq 1$ , we denote by  $\frac{p_n(x)}{q_n(x)} := [a_1(x), a_2(x), \dots, a_n(x)]$  the  $n$ -th *convergent* of the continued fraction expansion of  $x$ , where  $p_n(x)$  and  $q_n(x)$  are relatively prime. Clearly these convergents are rational numbers and  $p_n(x)/q_n(x) \rightarrow x$  as  $n \rightarrow \infty$  for all  $x \in [0, 1)$ . For more details about continued fractions, we refer the reader to a monograph of Khintchine [18].

A natural question is whether there exists some relationship between different expansions of some real number, for instance, its  $\beta$ -expansion and continued fraction expansion. For any irrational number  $x \in [0, 1)$  and  $n \geq 1$ , we denote by  $k_n(x)$  the exact number of partial quotients in the continued fraction expansion of  $x$  given by the first  $n$  digits in the  $\beta$ -expansion of  $x$ . In other words,

$$k_n(x) = \sup \{m \geq 0 : J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \subseteq I(a_1(x), \dots, a_m(x))\},$$

where  $J(\varepsilon_1(x), \dots, \varepsilon_n(x))$  and  $I(a_1(x), \dots, a_m(x))$  are called the *cylinders* of  $\beta$ -expansion and continued fraction expansion respectively (see Section 2 for the definition of the cylinder). It is easy to check that

$$0 \leq k_1(x) \leq k_2(x) \leq \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} k_n(x) = \infty.$$

The quantity  $k_n(x)$  was first introduced by Lochs [21] for  $\beta = 10$  and has been extensively studied by many mathematicians, see [2, 8, 9, 13, 20, 29, 30]. Applying the result of Dajani and Fieldsteel [6] (Theorem 5) to  $\beta$ -expansion and continued fraction expansion, Li and Wu [20] obtained a metric result of  $\{k_n, n \geq 1\}$ , that is, for  $\lambda$ -almost all  $x \in [0, 1)$ ,

$$\lim_{n \rightarrow \infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta}{\pi^2}. \quad (1.4)$$

The formula (1.4) has been stated for  $\beta = 10$  by a pioneering result of Lochs [21]. Barreira and Iommi [2] proved that the irregular set of points  $x \in [0, 1)$  for which the limit in (1.4) does not exist has full Hausdorff dimension. Li and Wu [20] gave some asymptotic results of  $k_n(x)/n$  for any irrational  $x \in [0, 1)$  not just a kind of almost all result (see also Wu [29]). For the special case  $\beta = 10$ , some limit theorems of  $\{k_n, n \geq 1\}$  were studied in the earlier literature. For instance, using Ruelle-Mayer operator, Faivre [8] showed an error term: for positive  $\varepsilon$  the Lebesgue measure of the set of  $x$ 's for which  $k_n(x)/n$  is more than  $\varepsilon$  away from  $(6 \log 2 \log 10)/\pi^2$  tends geometrically to zero. Later, he also proved a central limit theorem for the sequence  $\{k_n, n \geq 1\}$  in [9]. The law of the iterated logarithm for the sequence  $\{k_n, n \geq 1\}$  was established by Wu [30].

We wonder if the similar limit theorems of the sequence  $\{k_n, n \geq 1\}$  are still valid for general  $\beta > 1$ . It is worth pointing out that the lengths of cylinders play an important role in the study of  $\beta$ -expansion (see [5, 12]). The methods of Faivre (see [8, 9]) and Wu (see [29, 30]) rely heavily on the length of a cylinder for  $\beta = 10$ . Indeed, the  $n$ -th cylinder is a regular interval and its length equals always to  $10^{-n}$  for  $\beta = 10$ . For the general case  $\beta > 1$ , it is well-known that the  $n$ -th cylinder is a left-closed and right-open interval and its length has an absolute upper bound  $\beta^{-n}$ . Fan and Wang [12] obtained that the growth of the lengths of cylinders is multifractal and that the multifractal spectrum depends on  $\beta$ . However, for some “bad”  $\beta > 1$ , the  $n$ -th cylinder is irregular and there is no nontrivial absolute lower bound for its length, which can be much smaller than  $\beta^{-n}$ . This is the main difficulty we met. In fact, the authors have established a lower bound (not necessarily absolute) of the length of a cylinder (see [13, Proposition 2.3]) and obtained the central limit theorem and law of the iterated logarithm of  $\{k_n, n \geq 1\}$  for any  $\beta > 1$  in [13]. In the present paper, we make use of this lower bound to extend the result of Faivre [8] from  $\beta = 10$  to any  $\beta > 1$ , which indicates that the Lebesgue measure of the set of  $x \in [0, 1)$  for which  $k_n(x)/n$  deviates away from  $(6 \log 2 \log \beta)/\pi^2$  tends to 0 exponentially as  $n$  goes to  $\infty$ .

**Theorem 1.1.** *Let  $\beta > 1$  be a real number. For any  $\varepsilon > 0$ , there exist two positive constants  $A$  and  $\alpha$  (both depending on  $\beta$  and  $\varepsilon$ ) such that for all  $n \geq 1$ ,*

$$\lambda \left\{ x \in [0, 1) : \left| \frac{k_n}{n} - \frac{6 \log 2 \log \beta}{\pi^2} \right| \geq \varepsilon \right\} \leq A e^{-\alpha n}.$$

*Remark 1.2.* The above theorem immediately yields that for all  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} \lambda \left\{ x \in [0, 1) : \left| \frac{k_n}{n} - \frac{6 \log 2 \log \beta}{\pi^2} \right| \geq \varepsilon \right\} < +\infty.$$

That is,  $k_n(x)/n$  converges completely to  $(6 \log 2 \log \beta)/\pi^2$  (the definition of the complete convergence see [16]). This kind of convergence is stronger than almost everywhere convergence and sometimes more convenient to establish. By Borel-Cantelli lemma, we easily obtain the limit (1.4) for  $\lambda$ -almost all  $x \in [0, 1)$ .

For any  $x \in [0, 1)$ , we denote the partial sums of the form (1.1) by

$$x_n = \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \cdots + \frac{\varepsilon_n(x)}{\beta^n}$$

and call them the *convergents* of the  $\beta$ -expansion of  $x$ . It is clear that the sequence  $\{x_n, n \geq 1\}$  converges to  $x$  as  $n \rightarrow \infty$  for all  $x \in [0, 1)$ . If  $x - x_n > |x - p_n/q_n|$ , we say that the approximation of  $x$  by  $p_n/q_n$  is better than the approximation by  $x_n$  ( $n \in \mathbb{N}$ ). The formula (1.4) implies that for  $\lambda$ -almost all  $x \in [0, 1)$ , the larger  $\beta$  is (that is, the more symbols we use to code number  $x$ ), the more information about the continued fraction expansion we can obtained from its  $\beta$ -expansion. In other words, for sufficient large  $\beta > 1$ , the approximation of a real number by  $\beta$ -expansion is better than the approximation by continued fractions. More precisely, we show that the Lebesgue measure of the set for which the first  $n$  partial quotients of continued fraction expansion provide a better approximation for  $x$  than the first  $n$  digits of  $\beta$ -expansion decreases to 0 geometrically as  $n$  tends to  $\infty$  if  $\log \beta > \pi^2/(6 \log 2)$  and the case is converse when  $\log \beta < \pi^2/(6 \log 2)$ . Besides, we can also

see that the decay rates are related to the multifractal analysis for the Lyapunov exponent of the Gauss transformation (see Remark 3.6).

**Theorem 1.3.** *Let  $\beta > 1$  be a real number.*

(i) *If  $\log \beta > \pi^2/(6 \log 2)$ , then there exist two constants  $B_1 > 0$  and  $\gamma_1 > 0$  (both only depending on  $\beta$ ) such that for all  $n \geq 1$ ,*

$$\lambda \left\{ x \in [0, 1) : \left| x - \frac{p_n}{q_n} \right| \leq x - x_n \right\} \leq B_1 e^{-\gamma_1 n}.$$

(ii) *If  $\log \beta < \pi^2/(6 \log 2)$ , then there exist two constants  $B_2 > 0$  and  $\gamma_2 > 0$  (both only depending on  $\beta$ ) such that for all  $n \geq 1$ ,*

$$\lambda \left\{ x \in [0, 1) : x - x_n \leq \left| x - \frac{p_n}{q_n} \right| \right\} \leq B_2 e^{-\gamma_2 n}.$$

By Borel-Cantelli lemma, we immediately obtain the following corollary.

*Corollary 1.4.* Let  $\beta > 1$  be a real number.

(i) If  $\log \beta > \pi^2/(6 \log 2)$ , then for  $\lambda$ -almost all  $x \in [0, 1)$ , there exists positive integer  $N_1$  (depending on  $x$ ) such that for all  $n \geq N_1$ ,

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| > x - x_n.$$

(ii) If  $\log \beta < \pi^2/(6 \log 2)$ , then for  $\lambda$ -almost all  $x \in [0, 1)$ , there exists positive integer  $N_2$  (depending on  $x$ ) such that for all  $n \geq N_2$ ,

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| < x - x_n.$$

*Remark 1.5.* Since  $\log 10 < \pi^2/(6 \log 2)$ , we know that the approximation of some real number by decimal expansion (i.e.,  $\beta = 10$ ) is not better than the approximation by continued fraction expansion in the view of almost everywhere. This result was obtained by Faivre [8] in 1997. However, for the critical value  $\beta = \exp(\pi^2/(6 \log 2)) \approx 10.731$ , our methods are invalid and we do not know which is the better approximation by  $\beta$ -expansion and continued fraction expansion.

## 2. PRELIMINARY

This section is devoted to recalling some definitions and basic properties of the  $\beta$ -expansion and continued fraction expansion.

**2.1.  $\beta$ -expansions.** We first state some notions and basic properties of  $\beta$ -expansion.

**Definition 2.1.** An  $n$ -block  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is said to be admissible for  $\beta$ -expansions if there exists  $x \in [0, 1)$  such that  $\varepsilon_i(x) = \varepsilon_i$  for all  $1 \leq i \leq n$ . An infinite sequence  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots)$  is admissible if  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is admissible for all  $n \geq 1$ .

We denote by  $\Sigma_\beta^n$  the collection of all admissible sequences of length  $n$ . The following result of Rényi [26] implies that the dynamical system  $([0, 1), T_\beta)$  admits  $\log \beta$  as its topological entropy.

**Proposition 2.1** ([26]). *Let  $\beta > 1$ . For any  $n \geq 1$ ,*

$$\beta^n \leq \#\Sigma_\beta^n \leq \beta^{n+1}/(\beta - 1),$$

*where  $\#$  denotes the number of elements of a finite set.*

**Definition 2.2.** Let  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \Sigma_\beta^n$ . We define

$$J(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \{x \in [0, 1) : \varepsilon_i(x) = \varepsilon_i \text{ for all } 1 \leq i \leq n\}$$

and call it the  $n$ -th cylinder of  $\beta$ -expansion. In other words, it is the set of points beginning with  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  in their  $\beta$ -expansions. For any real number  $x \in [0, 1)$ ,  $J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))$  is said to be the  $n$ -th cylinder containing  $x$ .

*Remark 2.3.* The cylinder  $J(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is a left-closed and right-open interval with left endpoint

$$\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \dots + \frac{\varepsilon_n}{\beta^n}.$$

Moreover, the length of  $J(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  satisfies  $|J(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)| \leq 1/\beta^n$ .

For any  $x \in [0, 1)$  and  $n \geq 1$ , we assume that  $(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x), \dots)$  is the  $\beta$ -expansion of  $x$  and define

$$l_n(x) = \sup \{k \geq 0 : \varepsilon_{n+j}(x) = 0 \text{ for all } 1 \leq j \leq k\}. \quad (2.1)$$

That is, the length of the longest string of zeros just after the  $n$ -th digit in the  $\beta$ -expansion of  $x$ . The following proposition gives a lower bound, which is not absolute and related to  $l_n(x)$ .

**Proposition 2.2** ([13]). *Let  $\beta > 1$ . Then for any  $x \in [0, 1)$  and  $n \geq 1$ ,*

$$\frac{1}{\beta^{n+l_n(x)+1}} \leq |J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))| \leq \frac{1}{\beta^n},$$

where  $l_n(x)$  is defined as (2.1).

*Proof.* For any  $x \in [0, 1)$  and  $n \geq 1$ , we know that  $J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))$  is a left-closed and right-open interval with left endpoint

$$\omega_n(x) := \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \dots + \frac{\varepsilon_n(x)}{\beta^n}$$

and its length satisfies  $|J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))| \leq 1/\beta^n$ .

Since  $x \in J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))$ , we have that

$$\frac{1}{\beta^{n+l_n(x)+1}} \leq x - \omega_n(x) \leq |J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))|,$$

where the first inequality follows from  $\varepsilon_{n+1}(x) = \dots = \varepsilon_{n+l_n(x)}(x) = 0$  and  $\varepsilon_{n+l_n(x)+1}(x) \geq 1$  by the definition of  $l_n(x)$  in (2.1). This completes the proof.  $\square$

**2.2. Continued fractions.** Let us now introduce some elementary properties of continued fractions. For any irrational number  $x \in [0, 1)$  and  $n \geq 1$ , with the conventions  $p_{-1} = 1$ ,  $q_{-1} = 0$ ,  $p_0 = 0$ ,  $q_0 = 1$ , the quantities  $p_n$  and  $q_n$  satisfy the following recursive formula:

$$p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x) \quad \text{and} \quad q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x). \quad (2.2)$$

By the above recursive formula of  $p_n$  and  $q_n$ , we can easily obtain that

$$\frac{1}{2q_{n+1}^2(x)} \leq \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n^2(x)}. \quad (2.3)$$

This is to say that the speed of  $p_n(x)/q_n(x)$  approximating to  $x$  is dominated by  $q_n^{-2}(x)$ . So the denominator of the  $n$ -th convergent  $q_n(x)$  plays an important role in the problem of Diophantine approximation.

**Definition 2.4.** For any  $n \geq 1$  and  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , we call

$$I(a_1, \dots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}$$

the  $n$ -th cylinder of continued fraction expansion. For any real number  $x \in [0, 1)$ ,  $I(a_1(x), a_2(x), \dots, a_n(x))$  is said to be the  $n$ -th cylinder containing  $x$ .

The following proposition is about the structure and length of cylinders.

**Proposition 2.3** ([7]). *Let  $a_1, a_2, \dots, a_n \in \mathbb{N}$ . Then  $I(a_1, \dots, a_n)$  is an interval with two endpoints*

$$\frac{p_n}{q_n} \quad \text{and} \quad \frac{p_n + p_{n-1}}{q_n + q_{n-1}}$$

and the length of  $I(a_1, \dots, a_n)$  satisfies

$$\frac{1}{2q_n^2} \leq |I(a_1, \dots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}, \quad (2.4)$$

where  $p_n$  and  $q_n$  satisfy the recursive formula (2.2).

### 3. PROOFS OF THEOREMS

In the following, we denote by  $\mathbb{I}$  the set of all irrational numbers in  $[0, 1)$  and use the notation  $E(\xi)$  to denote the expectation of a random variable  $\xi$  w.r.t. the Lebesgue measure  $\lambda$ . For any  $\theta > 1/2$ , we define the so called *Diophantine pressure function* (see Kesseböhmer and Stratmann [17]) as

$$P(\theta) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{a_1, \dots, a_n} q_n^{-2\theta}([a_1, \dots, a_n]).$$

Kesseböhmer and Stratmann [17] proved that the Diophantine pressure function has a singularity at  $1/2$  and is decreasing, convex and real-analytic on  $(1/2, +\infty)$  satisfying

$$P(1) = 0 \quad \text{and} \quad P'(1) = -\pi^2/(6 \log 2). \quad (3.1)$$

Furthermore, they also studied the multifractal analysis for the Lyapunov exponent of the Gauss transformation  $T$  by using this function (see also Pollicott and Weiss [25] and Fan et al. [10] and [11]). More detailed analysis of this function can be founded in Mayer [22]. The following lemma establishes the relation between this function and the growth of the expectation of  $q_n$ , which plays an important role in our proofs.

**Lemma 3.1.** *For any  $\theta < 1/2$ ,*

$$P(1 - \theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E(q_n^{2\theta}).$$

*Proof.* By the definition of expectation, we know that

$$E(q_n^{2\theta}) = \sum_{a_1, \dots, a_n} q_n^{2\theta}([a_1, \dots, a_n]) \cdot \lambda(I(a_1, \dots, a_n)), \quad (3.2)$$

where  $a_1, \dots, a_n$  run over all the positive integers. In view of (2.4), we have that

$$\frac{1}{2q_n^2([a_1, \dots, a_n])} \leq \lambda(I(a_1, \dots, a_n)) \leq \frac{1}{q_n^2([a_1, \dots, a_n])}.$$

Combing this with (3.2), we deduce that

$$\frac{1}{2} \cdot \sum_{a_1, \dots, a_n} q_n^{-2(1-\theta)}([a_1, \dots, a_n]) \leq E(q_n^\theta) \leq \sum_{a_1, \dots, a_n} q_n^{-2(1-\theta)}([a_1, \dots, a_n])$$

and hence that

$$P(1-\theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E(q_n^{2\theta}) \quad \text{for any } \theta < 1/2.$$

□

**3.1. Proof of Theorem 1.1.** Recall that

$$k_n(x) = \sup \{m \geq 0 : J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \subseteq I(a_1(x), \dots, a_m(x))\},$$

we have the following lemma.

**Lemma 3.2.** *Let  $m \geq 1$  be an integer. Then*

$$\{x \in \mathbb{I} : k_n(x) \geq m\} = \{x \in \mathbb{I} : J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \subseteq I(a_1(x), \dots, a_m(x))\}.$$

To prove Theorem 1.1, we will show a stronger result.

**Proposition 3.1.** *Let  $a = (6 \log 2 \log \beta)/\pi^2$ . Then for any  $\varepsilon > 0$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} \leq \theta_1(\varepsilon) \quad (3.3)$$

with

$$\theta_1(\varepsilon) = \inf_{t > 0} \left\{ \frac{1}{t+1} \left( t \log \beta + (a + \varepsilon) P(t+1) \right) \right\} < 0$$

and for any  $0 < \varepsilon < a$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \leq a - \varepsilon \right\} \leq \theta_2(\varepsilon) \quad (3.4)$$

with

$$\theta_2(\varepsilon) = \inf_{0 < t < 1/2} \left\{ -t \log \beta + (a - \varepsilon) P(1-t) \right\} < 0.$$

*Remark 3.3.* By the domain of the function  $P(\cdot)$ , we first remark that the quantities  $\theta_1(\varepsilon)$  and  $\theta_2(\varepsilon)$  can be rewritten as

$$\theta_1(\varepsilon) = \inf_{t > -1/2} \left\{ \frac{1}{t+1} \left( t \log \beta + (a + \varepsilon) P(t+1) \right) \right\}$$

and

$$\theta_2(\varepsilon) = \inf_{t < 1/2} \left\{ -t \log \beta + (a - \varepsilon) P(1-t) \right\}.$$

In fact, for any  $\varepsilon > 0$ , we define  $f(t) = t \log \beta + (a + \varepsilon) P(t+1)$  for all  $-1/2 < t \leq 0$  and  $g(t) = -t \log \beta + (a - \varepsilon) P(1-t)$  for all  $t \leq 0$ . Moreover, we actually obtained that  $f(t) \geq 0$  and  $g(t) \geq 0$ . Since  $P(\cdot)$  is convex and real-analytic on  $(1/2, +\infty)$ , we know that  $P'(t+1) \leq P'(1)$  for any  $-1/2 < t \leq 0$  and  $P'(1-t) \geq P'(1)$  for any  $t \leq 0$ . It follows from (3.1) that

$$f'(t) = \log \beta + (a + \varepsilon) P'(t+1) \leq \log \beta + (a + \varepsilon) P'(1) = -\varepsilon \pi^2 / (6 \log 2) < 0$$

for any  $-1/2 < t \leq 0$  and

$$g'(t) = -\log \beta - (a - \varepsilon) P'(1-t) \leq -\log \beta - (a - \varepsilon) P'(1) = -\varepsilon \pi^2 / (6 \log 2) < 0$$

for any  $t \leq 0$ . Therefore,  $f$  is decreasing on  $(-1/2, 0]$  and  $g$  is decreasing on  $(-\infty, 0]$ . In view of (3.1), we obtain that  $f(t) \geq f(0) = 0$  when  $-1/2 < t \leq 0$  and  $g(t) \geq g(0) = 0$  if  $t \leq 0$ . Thus,  $\theta_1(\varepsilon)$  and  $\theta_2(\varepsilon)$  are established by the above formulas. Next we give a little more information about  $\theta_1(\varepsilon)$ . That is,

$$\theta_1(\varepsilon) \geq \inf_{t > -1/2} \{t \log \beta + (a + \varepsilon)P(t + 1)\}.$$

In fact, it is easy to check that  $\theta_1(\varepsilon) \geq \inf_{t > 0} \{t \log \beta + (a + \varepsilon)P(t + 1)\}$  since they are both negative. Moreover, the first remark has indicated that the infimum can take over all  $t > -1/2$ .

The following is a key lemma in the proof of the inequality (3.3).

**Lemma 3.4.** *Let  $\beta > 1$  be a real number and  $i \geq 0$  be an integer. Then for any  $n \geq 1$ ,*

$$\lambda \{x \in [0, 1) : l_n(x) \geq i\} \leq \frac{\beta^{1-i}}{\beta - 1}.$$

*Proof.* By the definition of  $l_n(x)$  in (2.1), it is clear to see that the result is true for  $i = 0$ . Now let  $i \geq 1$  be an integer. Then the set  $\{x \in [0, 1) : l_n(x) \geq i\}$  is the union of the  $(n + i)$ -th cylinders  $J(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_i)$ , where  $(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_\beta^n$ .

That is,

$$\{x \in [0, 1) : l_n(x) \geq i\} = \bigcup_{(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_\beta^n} J(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_i).$$

Since the cylinders  $J(\varepsilon_1, \dots, \varepsilon_n)$  and  $J(\varepsilon'_1, \dots, \varepsilon'_n)$  are disjoint for any  $(\varepsilon_1, \dots, \varepsilon_n) \neq (\varepsilon'_1, \dots, \varepsilon'_n) \in \Sigma_\beta^n$  and the length of  $J(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_i)$  always satisfies

$$|J(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_i)| \leq \frac{1}{\beta^{n+i}}$$

for any  $(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_\beta^n$ , it is from Proposition 2.1 that

$$\begin{aligned} \lambda \{x \in [0, 1) : l_n(x) \geq i\} &= \sum_{(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_\beta^n} |J(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_i)| \\ &\leq \#\Sigma_\beta^n \cdot \frac{1}{\beta^{n+i}} \leq \frac{\beta^{n+1}}{\beta - 1} \cdot \frac{1}{\beta^{n+i}} = \frac{\beta^{1-i}}{\beta - 1}. \end{aligned}$$

□

Now we are going to give the proof of (3.3).

*Proof of (3.3).* For any  $x \in [0, 1)$  and  $n \geq 1$ , Proposition 2.2 shows that

$$|J(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))| \geq \frac{1}{\beta^{n+l_n(x)+1}},$$

where  $l_n(x)$  is defined as (2.1). Let  $m \geq 1$  be an integer. In view of (2.4) and Lemma 3.2, we deduce that

$$\{x \in \mathbb{I} : k_n(x) \geq m\} \subseteq \left\{x \in \mathbb{I} : \frac{1}{\beta^{n+l_n(x)+1}} \leq \frac{1}{q_m^2(x)}\right\}. \quad (3.5)$$



Now we claim that for any  $\delta > 0$ ,

$$\left\{x \in \mathbb{I} : \frac{1}{\beta^{l_n(x)}} \leq \frac{\beta^{n+1}}{q_m^2(x)}\right\} \subseteq \left\{x \in \mathbb{I} : \frac{1}{\beta^{l_n(x)}} \leq \delta\right\} \cup \left\{x \in \mathbb{I} : \frac{\beta^{n+1}}{q_m^2(x)} \geq \delta\right\}.$$

Otherwise, if there exists some real number  $\delta_0 > 0$  such that  $1/(\beta^{l_n(x)}) > \delta_0$  and  $\beta^{n+1}/(q_m^2(x)) < \delta_0$  ( $x \in \mathbb{I}$ ), then we have  $1/(\beta^{l_n(x)}) > \beta^{n+1}/(q_m^2(x))$ . Combing this with (3.5), we obtain that

$$\lambda\{x \in \mathbb{I} : k_n(x) \geq m\} \leq \lambda\left\{x \in \mathbb{I} : \frac{1}{\beta^{l_n(x)}} \leq \delta\right\} + \lambda\left\{x \in \mathbb{I} : q_m^{-2}(x) \geq \frac{\delta}{\beta^{n+1}}\right\}. \quad (3.6)$$

Lemma 3.4 implies that

$$\lambda\left\{x \in \mathbb{I} : \frac{1}{\beta^{l_n(x)}} \leq \delta\right\} = \lambda\{x \in \mathbb{I} : l_n(x) \geq -\log_\beta \delta\} \leq C_\beta \delta,$$

where  $\log_\beta$  denotes the logarithm w.r.t. the base  $\beta$  and  $C_\beta = \beta^2/(\beta - 1)$ . For any  $t > 0$ , the Markov's inequality shows that

$$\lambda\left\{x \in \mathbb{I} : q_m^{-2}(x) \geq \frac{\delta}{\beta^{n+1}}\right\} = \lambda\left\{x \in \mathbb{I} : q_m^{-2t}(x) \geq \left(\frac{\delta}{\beta^{n+1}}\right)^t\right\} \leq \frac{\beta^{t(n+1)} \mathbb{E}(q_m^{-2t})}{\delta^t}.$$

Combining this with (3.6), we have that

$$\lambda\{x \in \mathbb{I} : k_n(x) \geq m\} \leq C_\beta \delta + \frac{\beta^{t(n+1)} \mathbb{E}(q_m^{-2t})}{\delta^t}.$$

It follows from Lemma 3.1 that

$$P(t+1) = \lim_{m \rightarrow \infty} \frac{1}{m} \log \mathbb{E}(q_m^{-2t}).$$

Hence, for any  $\eta > 0$ , there exists a positive number  $M$  (depending on  $\eta$ ) such that for all  $m \geq M$ , we have

$$\mathbb{E}(q_m^{-2t}) \leq e^{m(P(t+1)+\eta)}.$$

Therefore, for any  $m \geq M$ , we obtain that

$$\lambda\{x \in \mathbb{I} : k_n(x) \geq m\} \leq C_\beta \delta + \frac{\beta^{t(n+1)} e^{m(P(t+1)+\eta)}}{\delta^t}. \quad (3.7)$$

For any  $\varepsilon > 0$  and  $n \geq 1$ , let  $m_n = [n(a + \varepsilon)]$ . Then  $m_n \rightarrow \infty$ ,  $m_n/n \leq a + \varepsilon$  and  $m_n/n \rightarrow a + \varepsilon$  as  $n \rightarrow \infty$ . So, there exists a positive number  $N$  (depending on  $\eta$  and  $\varepsilon$ ) such that for all  $n \geq N$ , we have that  $m_n \geq M$ . Fixed such  $n \geq N$ , it follows from (3.7) that

$$\lambda\{x \in \mathbb{I} : k_n(x) \geq m_n\} \leq C_\beta \delta + \frac{\beta^{t(n+1)} e^{m_n(P(t+1)+\eta)}}{\delta^t}.$$

Now we choose a suitable  $\delta > 0$  such that  $f(\delta) = C_\beta \delta + \delta^{-t} \beta^{t(n+1)} e^{m_n(P(t+1)+\eta)}$  reaches the minimum value. To do this, letting the derivative of  $f(\delta)$  equals to zero, we calculate that  $\delta = \left(C_\beta^{-1} t \beta^{t(n+1)} e^{m_n(P(t+1)+\eta)}\right)^{\frac{1}{t+1}}$ . Thus we deduce that

$$\lambda\{x \in \mathbb{I} : k_n(x) \geq m_n\} \leq H(t, \beta) \cdot \left(\beta^{t(n+1)} e^{m_n(P(t+1)+\eta)}\right)^{\frac{1}{t+1}}, \quad (3.8)$$

where  $H(t, \beta) = \left(C_\beta^t \cdot t\right)^{\frac{1}{t+1}} (1 + t^{-1})$  is a constant only depending on  $t$  and  $\beta$ . Since  $m_n/n \leq a + \varepsilon$ , we obtain that

$$\lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} \leq \lambda \{ x \in \mathbb{I} : k_n(x) \geq m_n \}.$$

Combing this with (3.8), we have that

$$\lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} \leq H(t, \beta) \cdot \left( \beta^{t(n+1)} e^{m_n(P(t+1)+\eta)} \right)^{\frac{1}{t+1}}.$$

Notice that  $m_n/n \rightarrow a + \varepsilon$  as  $n \rightarrow \infty$ , we know that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} \leq \frac{1}{t+1} \left( t \log \beta + (a + \varepsilon) (P(t+1) + \eta) \right)$$

and hence that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} \leq \frac{1}{t+1} \left( t \log \beta + (a + \varepsilon) P(t+1) \right)$$

holds for any  $t > 0$  since  $\eta > 0$  is arbitrary.

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} \leq \theta_1(\varepsilon)$$

with

$$\theta_1(\varepsilon) = \inf_{t > 0} \left\{ \frac{1}{t+1} \left( t \log \beta + (a + \varepsilon) P(t+1) \right) \right\}.$$

Now it remains to prove that  $\theta_1(\varepsilon) < 0$ . Let  $h(\omega)$  be the function defined as

$$h(\omega) = \omega \log \beta + (a + \varepsilon) P(\omega + 1) \quad \text{for all } \omega > -1/2.$$

It is clear to see  $h$  is real-analytic on  $(-1/2, +\infty)$  since the pressure function  $P(\cdot)$  is real-analytic. By the properties of  $P(\cdot)$  in (3.1), we know that  $h(0) = 0$  and  $h'(0) = -\pi^2 \varepsilon / (6 \log 2) < 0$ . Hence there exists  $t_0 > 0$  such that  $h(t_0) < 0$  by the definition of derivative. Thus,  $\theta_1(\varepsilon) \leq h(t_0) < 0$ .  $\square$

To prove the inequality (3.4), we need the following lemma whose proof is inspired by Wu [29] (see also Li and Wu [20]).

**Lemma 3.5.** *Let  $n \geq 1$ ,  $i \geq 1$  be integers and  $x \in [0, 1)$  be an irrational number such that  $J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \not\subseteq I(a_1(x), \dots, a_{i+1}(x))$ . Then*

$$\frac{1}{6q_{i+3}^2(x)} \leq |J(\varepsilon_1(x), \dots, \varepsilon_n(x))| \leq \frac{1}{\beta^n}.$$

*Proof.* Since  $J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \not\subseteq I(a_1(x), \dots, a_{i+1}(x))$ , we know that at least one endpoint of  $J(\varepsilon_1(x), \dots, \varepsilon_n(x))$  does not belong to  $I(a_1(x), \dots, a_{i+1}(x))$ . Without loss of generality, we assume that the right endpoint of  $J(\varepsilon_1(x), \dots, \varepsilon_n(x))$  does not belong to  $I(a_1(x), \dots, a_{i+1}(x))$ , i.e., the right endpoint of  $I(a_1(x), \dots, a_{i+1}(x))$  belongs to  $J(\varepsilon_1(x), \dots, \varepsilon_n(x))$ .

**Case I.** If  $i$  is even, we know that  $I(a_1(x), \dots, a_{i+1}(x))$  is decomposed into a countable  $(i+2)$ -th cylinders like  $I(a_1(x), \dots, a_{i+1}(x), j)$  ( $j \in \mathbb{N}$ ) and these cylinders  $I(a_1(x), \dots, a_{i+1}(x), 1), I(a_1(x), \dots, a_{i+1}(x), 2), \dots$  run from left to right. Since  $x \in J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \cap I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x))$ , we have that

$$I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x) + 1) \subseteq J(\varepsilon_1(x), \dots, \varepsilon_n(x)).$$

By (2.2) and (2.4), we deduce that

$$|I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x) + 1)| \geq \frac{1}{6q_{i+2}^2(x)} \geq \frac{1}{6q_{i+3}^2(x)}.$$

Hence, in view of Proposition 2.2, we obtain

$$\frac{1}{6q_{i+3}^2(x)} \leq |J(\varepsilon_1(x), \dots, \varepsilon_n(x))| \leq \frac{1}{\beta^n}.$$

**Case II.** If  $i$  is odd, we consider the  $(i+2)$ -th cylinder  $I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x))$ . We know that  $I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x))$  can be decomposed into a countable  $(i+3)$ -th cylinders like  $I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x), j)$  ( $j \in \mathbb{N}$ ) and these cylinders  $I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x), 1), I(a_1(x), \dots, a_{i+1}(x), a_{i+2}(x), 2), \dots$  also run from left to right. Notice that  $x \in J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \cap I(a_1(x), \dots, a_{i+2}(x), a_{i+3}(x))$ , we obtain that  $I(a_1(x), \dots, a_{i+1}(x), a_{i+3}(x) + 1) \subseteq J(\varepsilon_1(x), \dots, \varepsilon_n(x))$ . Therefore,

$$|J(\varepsilon_1(x), \dots, \varepsilon_n(x))| \geq \frac{1}{6q_{i+3}^2(x)}.$$

By Proposition 2.2, we complete the proof.  $\square$

Now we are ready to prove (3.4).

*Proof of (3.4).* Let  $m \geq 1$  be an integer. By Lemma 3.2, we obtain that

$$\{x \in \mathbb{I} : k_n(x) \leq m\} = \{x \in \mathbb{I} : J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \not\subseteq I(a_1(x), \dots, a_{m+1}(x))\}.$$

It follows from Lemma 3.5 that

$$\{x \in \mathbb{I} : J(\varepsilon_1(x), \dots, \varepsilon_n(x)) \not\subseteq I(a_1(x), \dots, a_{m+1}(x))\} \subseteq \left\{x \in \mathbb{I} : \frac{1}{6q_{m+3}^2(x)} \leq \frac{1}{\beta^n}\right\}.$$

Therefore,

$$\lambda\{x \in \mathbb{I} : k_n(x) \leq m\} \leq \lambda\left\{x \in \mathbb{I} : q_{m+3}^2(x) \geq \frac{\beta^n}{6}\right\}.$$

For any  $0 < t < 1/2$ , the Markov's inequality yields that

$$\lambda\{x \in \mathbb{I} : k_n(x) \leq m\} \leq \lambda\left\{x \in \mathbb{I} : q_{m+3}^{2t}(x) \geq \left(\frac{\beta^n}{6}\right)^t\right\} \leq \frac{6^t \mathbb{E}(q_{m+3}^{2t})}{\beta^{tn}}. \quad (3.9)$$

By Lemma 3.1, for any  $\eta > 0$ , there exists a positive number  $M$  (depending on  $\eta$ ) such that for all  $m \geq M$ , we have

$$\mathbb{E}(q_{m+3}^{2t}) \leq e^{m(P(1-t)+\eta)}.$$

Combing this with (3.9), for any  $m \geq M$ , we obtain that

$$\lambda\{x \in \mathbb{I} : k_n(x) \leq m\} \leq \frac{6^t e^{m(P(1-t)+\eta)}}{\beta^{tn}}. \quad (3.10)$$

For any  $0 < \varepsilon < a$  and  $n \geq 1$ , let  $m_n = [n(a-\varepsilon)] + 1$ . Then  $m_n \rightarrow \infty$ ,  $m_n/n \geq a-\varepsilon$  and  $m_n/n \rightarrow a-\varepsilon$  as  $n \rightarrow \infty$ . So, there exists a positive number  $N$  (depending on  $\eta$  and  $\varepsilon$ ) such that for all  $n \geq N$ , we have that  $m_n \geq M$ . Now we fix such  $n \geq N$ , in view of (3.10), we deduce that

$$\lambda\{x \in \mathbb{I} : k_n(x) \leq m_n\} \leq 6^t \beta^{-tn} e^{m_n(P(1-t)+\eta)}.$$

Notice that  $m_n/n \geq a - \varepsilon$ , we have

$$\lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \leq a - \varepsilon \right\} \leq \lambda \{ x \in \mathbb{I} : k_n(x) \leq m_n \} \leq 6^t \beta^{-tn} e^{m_n(P(1-t)+\eta)}$$

and hence that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \leq a - \varepsilon \right\} \leq -t \log \beta + (a - \varepsilon)P(1 - t)$$

for any  $0 < t < 1/2$  since  $m_n/n \rightarrow a - \varepsilon$  as  $n \rightarrow \infty$  and  $\eta > 0$  is arbitrary. Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left( \frac{k_n}{n} \leq a - \varepsilon \right) \leq \theta_2(\varepsilon)$$

with

$$\theta_2(\varepsilon) = \inf_{0 < t < 1/2} \{ -t \log \beta + (a - \varepsilon)P(1 - t) \}.$$

Now we need to show that  $\theta_2(\varepsilon) < 0$ . For any  $\omega < 1/2$ , we consider the function

$$h(\omega) = -\omega \log \beta + (a - \varepsilon) \log P(1 - \omega).$$

Then it is easy to check that  $h$  is real-analytic on  $(-\infty, 1/2)$  and satisfies  $h(0) = 0$  and  $h'(0) = -\pi^2 \varepsilon / (6 \log 2) < 0$  because of the properties of  $P(\cdot)$  in (3.1). So, if  $t > 0$  sufficiently close to 0, we obtain that  $\theta_2(\varepsilon) < 0$ .  $\square$

We end this section with the proof of Theorem 1.1.

*Proof of Theorem 1.1.* For any  $\varepsilon > 0$  and  $n \geq 1$ , since

$$\begin{aligned} & \lambda \left\{ x \in \mathbb{I} : \left| \frac{k_n(x)}{n} - a \right| \geq \varepsilon \right\} \\ &= \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} + \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \leq a - \varepsilon \right\}, \end{aligned}$$

we obtain that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \left| \frac{k_n(x)}{n} - a \right| \geq \varepsilon \right\} \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left( \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \geq a + \varepsilon \right\} + \lambda \left\{ x \in \mathbb{I} : \frac{k_n(x)}{n} \leq a - \varepsilon \right\} \right) \\ &\leq \max\{\theta_1(\varepsilon), \theta_2(\varepsilon)\}, \end{aligned}$$

where the last inequality follows from the inequalities (3.3) and (3.4) and  $\theta_1(\varepsilon)$  and  $\theta_2(\varepsilon)$  are as defined in Proposition 3.1. Therefore, for any  $\varepsilon > 0$ , there exist positive real  $\alpha$  (only depending on  $\beta$  and  $\varepsilon$ ) and positive integer  $N$  such that for all  $n > N$ , we have

$$\lambda \left\{ x \in \mathbb{I} : \left| \frac{k_n(x)}{n} - a \right| \geq \varepsilon \right\} \leq e^{-\alpha n}. \quad (3.11)$$

For all  $1 \leq n \leq N$ , since the probabilities of the left-hand side in (3.11) are bounded, we can choose sufficiently large  $A$  (only depending on  $\beta$  and  $\varepsilon$ ) such that

$$\lambda \left\{ x \in \mathbb{I} : \left| \frac{k_n(x)}{n} - a \right| \geq \varepsilon \right\} \leq A e^{-\alpha n}$$

holds for all  $n \geq 1$ . Thus, we complete the proof of Theorem 1.1.  $\square$

**3.2. Proof of Theorem 1.3.** Being similar to the proof of Theorem 1.1, we give a stronger result than that of Theorem 1.3 as well.

**Proposition 3.2.** *Let  $\beta > 1$  be a real number.*

(i) *If  $\log \beta > \pi^2/(6 \log 2)$ , then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \left| x - \frac{p_n}{q_n} \right| \leq x - x_n \right\} \leq \theta \quad (3.12)$$

with

$$\theta = \inf_{0 < t < 1/2} \{ -t \log \beta + P(1-t) \} < 0.$$

(ii) *If  $\log \beta < \pi^2/(6 \log 2)$ , then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : x - x_n \leq \left| x - \frac{p_n}{q_n} \right| \right\} \leq \theta^* \quad (3.13)$$

with

$$\theta^* = \inf_{t > 0} \left\{ \frac{1}{t+1} (t \log \beta + P(t+1)) \right\} < 0.$$

*Remark 3.6.* By the similar methods with Remark 3.3, the constants  $\theta$  and  $\theta^*$  can also be rewritten as

$$\theta = \inf_{t < 1/2} \{ -t \log \beta + P(1-t) \}$$

and

$$\theta^* = \inf_{t > -1/2} \left\{ \frac{1}{t+1} (t \log \beta + P(t+1)) \right\}.$$

Besides, we can also give more remarks on  $\theta$  and  $\theta^*$ , which indicates that  $\theta$  and  $\theta^*$  are related to the multifractal analysis for the Lyapunov exponent of the Gauss transformation. Recall that Kesseböhmer and Stratmann [17, Theorem 1.3] (see also Fan et al. [10]) proved that

$$\tau(\gamma) := \dim_H \{ x \in [0, 1] : \mathcal{L}(x) = \gamma \} = \frac{\inf_{t \in \mathbb{R}} \{ t \cdot \gamma + P(t) \}}{\gamma}$$

for any  $\gamma \geq 2 \log((\sqrt{5} + 1)/2)$ , where  $\dim_H$  denotes the Hausdorff dimension and  $\mathcal{L}(x)$  is the Lyapunov exponent of the Gauss transformation  $T$  defined as

$$\mathcal{L}(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |(T^n)'(x)|,$$

if the limit exists. Since  $\log \beta > \pi^2/(6 \log 2) > 2 \log((\sqrt{5} + 1)/2)$ , a simple calculation implies that the constant  $\theta$  have an alternative form

$$\theta = (\tau(\log \beta) - 1) \log \beta.$$

It is not difficult to check that

$$\theta^* \geq \inf_{t > -1/2} \{ t \log \beta + P(t+1) \} = (\tau(\log \beta) - 1) \log \beta,$$

where the last equality only holds for  $\log \beta \geq 2 \log((\sqrt{5} + 1)/2)$ . By Theorem 1.3 of Kesseböhmer and Stratmann [17] (see also Theorem 1.3 of Fan et al. [10]), we know that  $-(\log \beta)/2 < \theta < 0$  for  $\log \beta > \pi^2/(6 \log 2)$  and  $\theta^* \geq -\log \beta$  for  $\log \beta \geq 2 \log((\sqrt{5} + 1)/2)$ . This is one way to show  $\theta$  is negative and also gives the lower bounds for  $\theta$  and  $\theta^*$ .

We first give the proof of the inequality (3.12).

*Proof of (3.12).* For any irrational number  $x \in [0, 1)$  and  $n \geq 1$ , in view of (1.1) and (2.3), we have that

$$x - x_n = \frac{T_\beta^n x}{\beta^n} \leq \frac{1}{\beta^n} \quad \text{and} \quad \left| x - \frac{p_n(x)}{q_n(x)} \right| \geq \frac{1}{2q_{n+1}^2(x)}.$$

Hence, we obtain that

$$\lambda \left\{ x \in \mathbb{I} : \left| x - \frac{p_n}{q_n} \right| \leq x - x_n \right\} \leq \lambda \left\{ x \in \mathbb{I} : \frac{1}{2q_{n+1}^2(x)} \leq \frac{1}{\beta^n} \right\}.$$

For any  $0 < t < 1/2$ , the Markov's inequality implies that

$$\lambda \left\{ x \in \mathbb{I} : \frac{1}{2q_{n+1}^2(x)} \leq \frac{1}{\beta^n} \right\} \leq \lambda \left\{ x \in \mathbb{I} : q_{n+1}^{2t}(x) \geq \left( \frac{\beta^n}{2} \right)^t \right\} \leq \frac{2^t \mathbb{E}(q_{n+1}^{2t})}{\beta^{tn}}.$$

The similar methods of (3.10) yield that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \left| x - \frac{p_n}{q_n} \right| \leq x - x_n \right\} \leq -t \log \beta + \mathbb{P}(1 - t)$$

for any  $0 < t < 1/2$ . Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \left| x - \frac{p_n}{q_n} \right| \leq x - x_n \right\} \leq \theta$$

with

$$\theta = \inf_{0 < t < 1/2} \{ -t \log \beta + \mathbb{P}(1 - t) \}.$$

The condition  $\log \beta > \pi^2/(6 \log 2)$  assures that  $\theta < 0$  using the similar techniques at the end of the proof of (3.4).  $\square$

Next we prove the inequality (3.13).

*Proof of (3.13).* For any irrational number  $x \in [0, 1)$  and  $n \geq 1$ , by (1.1) and (2.3), we know that

$$x - x_n = \frac{T_\beta^n x}{\beta^n} \quad \text{and} \quad \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n^2(x)}.$$

Thus

$$\left\{ x \in \mathbb{I} : x - x_n \leq \left| x - \frac{p_n}{q_n} \right| \right\} \leq \left\{ x \in \mathbb{I} : \frac{T_\beta^n x}{\beta^n} \leq \frac{1}{q_n^2(x)} \right\}. \quad (3.14)$$

Note that for any  $\delta > 0$ ,

$$\left\{ x \in \mathbb{I} : \frac{T_\beta^n x}{\beta^n} \leq \frac{1}{q_n^2(x)} \right\} \subseteq \{ x \in \mathbb{I} : T_\beta^n x \leq \delta \} \cup \left\{ x \in \mathbb{I} : \frac{\beta^n}{q_n^2(x)} \geq \delta \right\},$$

otherwise, if there exists some real number  $\delta_0 > 0$  such that  $T_\beta^n x > \delta_0$  and  $\beta^n/(q_n^2(x)) < \delta_0$  ( $x \in \mathbb{I}$ ), then we have  $T_\beta^n x > \beta^n/(q_n^2(x))$  and hence that  $T_\beta^n x/\beta^n > 1/(q_n^2(x))$ . Therefore,

$$\lambda \left\{ x \in \mathbb{I} : \frac{T_\beta^n x}{\beta^n} \leq \frac{1}{q_n^2(x)} \right\} \leq \lambda \{ x \in \mathbb{I} : T_\beta^n x \leq \delta \} + \lambda \left\{ x \in \mathbb{I} : q_n^{-2}(x) \geq \frac{\delta}{\beta^n} \right\}. \quad (3.15)$$

Since  $T_\beta$  is measure-preserving w.r.t.  $\mu$ , we have  $\mu\{x \in \mathbb{I} : T_\beta^n x \leq \delta\} = \delta$ . The relation (1.2) between  $\mu$  and  $\mathbb{P}$  yields that

$$\lambda \{ x \in \mathbb{I} : T_\beta^n x \leq \delta \} \leq C\delta,$$

where  $C > 1$  is a constant only depending on  $\beta$ . For any  $t > 0$ , the Markov's inequality indicates that

$$\lambda \left\{ x \in \mathbb{I} : q_n^{-2}(x) \geq \frac{\delta}{\beta^n} \right\} = \lambda \left\{ x \in \mathbb{I} : q_n^{-2t}(x) \geq \left( \frac{\delta}{\beta^n} \right)^t \right\} \leq \frac{\beta^{tn} \mathbf{E}(q_n^{-2t})}{\delta^t}.$$

Combining these with (3.14) and (3.15), we deduce that

$$\left\{ x \in \mathbb{I} : x - x_n \leq \left| x - \frac{p_n}{q_n} \right| \right\} \leq C\delta + \frac{\beta^{tn} \mathbf{E}(q_n^{-2t})}{\delta^t}.$$

Using similar methods of the proof of (3.7) and (3.8), we actually obtain that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : x - x_n \leq \left| x - \frac{p_n}{q_n} \right| \right\} \leq \frac{1}{t+1} (t \log \beta + \mathbf{P}(t+1))$$

for any  $t > 0$  and hence that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : x - x_n \leq \left| x - \frac{p_n}{q_n} \right| \right\} \leq \theta^*$$

with

$$\theta^* = \inf_{t > 0} \left\{ \frac{1}{t+1} (t \log \beta + \mathbf{P}(t+1)) \right\}.$$

The condition  $\log \beta < \pi^2/(6 \log 2)$  guarantees that  $\theta^* < 0$  by the similar techniques at the end of the proof of (3.3).  $\square$

At last, we are ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* In view of Proposition 3.2, the similar methods of the proof of Theorem 1.1 give the proofs of (i) and (ii) in Theorem 1.3.  $\square$

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